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# MATHEMATICAL MODEL OF THERMAL PROCESS WITH AN UNKNOWN SOURCE

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**Purpose.** The purpose of this work is to construct an algorithm for solving the problem of recovery of the external source of heat during thermocyclic and pulse treatments of moving and stationary wire and other products of cylindrical shape. **Methodology.** Mathematical models that are used in the study of physical and technological processes with the use of computer mathematics methods are powerful tools of modern research. In this paper, an example of a mathematical model of the thermal process that takes place in a moving medium, the inverse problem is proposed. **Originality** Heat sources are presented in the form of a finite function in the boundary conditions. The inverse problem is considered in extreme statement. The condition of the heat balance in integral form was used for the construction of the objective quadratic functional. This condition is more informational in the inverse heat conduction problem. The residual criterion is introduced and an iterative process by the method of conjugate gradients is constructed. **Results.** The solution of this problem allows to determine the temperature distribution and the density of the external sources of heat. The problem is solved numerically by constructing absolutely stable with respect to the initial data and the right hand side six-point difference scheme. Then, the sweep method is used for solving the system of equations. References 18, tables 0, figures 0.

**Key words:** mathematical model, inverse problem, heat sources, condition of the heat balance.

# МАТЕМАТИЧНА МОДЕЛЬ ТЕПЛОВОГО ПРОЦЕСУ ІЗ НЕВІДОМОЮ ФУНКЦІЄЮ ДЖЕРЕЛА

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Математичні моделі, що використовуються при вивченні фізичних і технологічних процесів із застосуванням методів комп’ютерної математики є потужним інструментом сучасних досліджень. У роботі, на прикладі математичної моделі теплового процесу, що відбувається у рухомому середовищі пропонується обернена задача, розв’язок якої дозволяє визначати температурний розподіл і щільність зовнішніх джерел тепла. Джерела тепла представлені у вигляді фінітної функції в крайових умовах. Обернена задача розглянута в екстремальній постановці. Для побудови цільового квадратичного функціоналу використовувалася умова теплового балансу в інтегральній формі. Ця ж умова є додатковою інформацією в оберненій задачі теплопровідності. Уводиться критерій нев’язки та будується ітераційний процес за методом спряжених градієнтів. Задача розв’язується чисельним методом шляхом побудови абсолютно стійкої, за початковими умовами та правою частиною, шеститочкової різницевої схеми. Для розв’язання системи алгебраїчних рівнянь використано метод прогону.

**Ключові слова:** математична модель, зворотна задача, джерело тепла, умова теплового балансу.

PROBLEM STATEMENT. In powder metallurgy in the production of wire, as well as other forms of products, the processes of heat treatment of moving and stationary objects are widely used. Here, heat treatment is used as a separate operation, or in combination with plastic deformation [1–3]. This is due to the requirements to the quality of the final product. Therefore, along with plastic deformation are various types of heat treatment by both external and internal heat sources [1–3]. Particularly important is the heat treatment, which allows to create physical and mechanical properties of metals and composite materials. Besides the usual heat treatment methods thermo cyclic and pulse treatments are used, which is particularly effective in obtaining ultrafine wire using electro-plastic deformation techniques. [1–3].

Thermal cycle processing is a process of the thermal affecting material, that takes place due to the continuous cyclic change of temperature under the action of external or internal sources of heat and is accompanied by multiple structural or phase changes in the material. Processing is performed in the gas or muffle furnace (first case) or by passing an electric current (second case) through the product or semi-finished product.

Papers [4–8] contain studies of the thermal processes that occur during sintering of powder materials, various kinds of heat treatment and thermal processes that occur during wire drawing. Mathematical models describe temperature distribution during processing of moving and stationary wire and other products of cylindrical shape.

Initial-boundary value problems for linear and quasi-linear heat equation in cylindrical coordinate system $(r, z, \phi, t)$ are considered as mathematical models. The special features of these models are in the fact that the tasks underlying them describe the thermal processes of moving and stationary environment with homogeneous and inhomogeneous heat conduction equations.

The purpose of this work is to construct an algorithm for solving the problem of recovery of the external source of heat during thermo cyclic and pulse treatments of moving and stationary wire and other products of cylindrical shape.

EXPERIMENTAL PART AND RESULTS OBTAINED. From a mathematical point of view, the study of the temperature distribution in moving and stationary axially symmetric objects can be carried out considering different initial-boundary value problems for linear and nonlinear heat equation by introducing certain restrictions on the equation and attracting the appropriate boundary conditions that characterize the physical characteristics of the heating process. Since most of the temperature distribution investigated by heating the product of cylindrical shape does not depend on the coordinates $\phi$, the partial derivative with respect to this variable in the heat equation can be neglected. Wires and other cylindrical shaped products are considered in the form of a moving or stationary cylindrical isotropic medium with constant parameters and thermal characteristics with the heating zone length *L*. In this case mathematical models of temperature fields in which there are both external and internal sources of heat are studied [5, 6].

In this work we investigate the mathematical model of the temperature field during thermal cycling treatment of products by external heat source. External sources caused by heat exchange of the product with the environment according to Newton's and Stefan-Boltzmann's laws. In the mathematical model external heat sources are represented as boundary conditions of the first, second or third kind with a finite function .

Influence of non-linear components in the equation and the boundary conditions on the temperature distribution is considered in mathematical models.

The solution of the boundary problem for the heat equation with an external periodically operating heat sources is considered.

In the mathematical model the external heat sources are represented in the form of boundary conditions of the first second or third kind. The mathematical model of the temperature field during pulsed treatment of cylindrical shape sample, has the form [7]

$$\lambda\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})+\lambda\frac{\partial^2 T}{\partial z^2}-vc\rho_n\frac{\partial T}{\partial z}-c\rho_n\frac{\partial T}{\partial t}=0,\quad (r,z)\in\Omega_t \quad (1)$$

$$T(r,z,0)=T_0, \quad (2)$$

$$T(r,0,t)=T_0,\qquad T(r,l,t)=T_l, \quad (3)$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=0}=0,\quad \lambda\left.\frac{\partial T}{\partial r}\right|_{r_0=0}=f_{12}(t)\left[-\alpha(T-T_c)-\varepsilon\sigma(T^4-T_c^4)\right], \quad (4)$$

where

$$f_{12}(t)=\begin{cases}\frac{t}{t_0}-2n, & 2nt_0\le t\le(2n+1)t_0\\ -\frac{t}{t_0}+2(n+1), & (2n+1)t_0<t\le(2n+2)t_0\end{cases},$$

$$f_{12}(t)=\begin{cases}\frac{2kt}{t_0}-2kn, & nt_0\le t\le(n+\frac{1}{2mk})t_0\\ \frac{1}{m}, & (n+\frac{1}{2mk})t_0<t\le(n+\frac{2m-1}{2mk})t_0\\ -\frac{2kt}{t_0}+2(kn+1), & (n+\frac{2m-1}{2mk})t_0<t\le(n+\frac{1}{k})t_0\\ 0, & (n+\frac{1}{k})t_0<t<(n+1)t_0\end{cases},$$

$$f_{12}(t)=\left|\sin\left(\frac{t}{t_0}\right)\right|,$$

where $\lambda, c, \rho_m$ are thermal conductivity, heat capacity and density, $T_c$ – ambient temperature, $\alpha,\varepsilon,\sigma$ – the convective heat transfer coefficient from the surface, the emissivity factor and the Stefan-Boltzmann constant. By using the relation [9].

$$u(z,t)=\frac{2}{r_0^2}\int_0^{r_0}T(r,z,t)\,rdr \quad (5)$$

and the boundary condition (3), we obtain a problem for the determination of the average temperature along the radius in the area $Q_T=\{(z,t)|\,z\in(0,l),t\in(0,t_0)\}$

$$\lambda\frac{\partial^2 u}{\partial^2 z}-v(t)c\rho_n\frac{\partial u}{\partial z}-c\rho_n\frac{\partial u}{\partial t}=\frac{2\varepsilon\sigma}{r_0}\left(T_c^4-u^4\right)+f_{12}(t)\frac{2}{r_0}\alpha(T_c-u), \quad (6)$$

$$u(z,0)=T_0, \quad (7)$$

$$u(0,t)=T_0,\qquad u(l,t)=T_l. \quad (8)$$

In the latter formula the heat exchange surface of the moving medium with the environment is kept in mind. The transformation (5) makes it possible to reduce the dimension of the problem and reduce it to a solution of the first boundary value problem for the quasilinear heat equation. Substituting the variables (9) in problem (6)–(8), we reduce Eq. (6) to the canonical form

$$u(z,t)=U(z,t)e^{\mu(t)z+\eta(t)t} \quad (9)$$

and we obtain the following problem in the area $\bar{Q}_T=\{(z,t)|\,z\in(0,l),t\in(0,t_0)\}$

$$\frac{\partial U}{\partial t}=a^2\frac{\partial^2 U}{\partial^2 z}+f(z,t),\qquad (z,t)\in Q_T \quad (10)$$

$$U(z,0)=T(z)=T_0e^{-\mu(0)z},\qquad 0\le z\le l \quad (11)$$

$$U(0,t)=T(t)=T_0 e^{-\eta(t)t}, \quad T(z)=T(t), \tag{12}$$

$$U(l,t)=T_1(t), \quad 0 \le t \le t_0,$$

where

$$f(z,t)=f_{12}(t)\left(\frac{-2\alpha T_c \pi^2}{\pi^2 r_0 c\rho_n}\right)e^{-\mu z-\eta(t)t}, \ \mu=\frac{vc\rho_n}{2\lambda},$$

$$\eta(t)=\frac{-v^2 c\rho_n}{4\lambda}+f_{12}(t)\frac{2\alpha}{r_0 c\rho_n}.$$

We have a correctly formulated problem, because the solution of problem (10)–(12) exists, it is unique and stable with respect to small perturbations $f(z,t)$, $T(z)$ $T_1(z)$.

Problem of reconstructing the heat source.

In the case where the heat source is a known function, we come to the inverse problem.

Such problems arise during control of temperature fields. Here it is necessary to find temperature distribution in the axisymmetric environment. Let the thermal characteristics of the medium be constant. After applying averaging (6) and further transformations the problem (1)–(4) is transformed into (10)–(12). Here the function $f(z,t)$ can be determined fully only when we know the temperature distribution throughout the heating area. Therefore, during assigning the heat sources we assume the amount of energy that turns into heating of the area and loss from the surface to be a known. The condition of the heat balance in the case of an external source takes the form

$$\begin{aligned}&\int\limits_{0+}^{t}\iint\limits_{G}\alpha(T)\frac{T(r,z,t)-T_c}{v}dg dt = \\ &= c\rho_n\int\limits_{0+}^{t_0}\int\limits_{0}^{r_0}\int\limits_{0}^{l} f_{12}(t)(T(r,z,t)-T_0)dzdrdt.\end{aligned} \tag{13}$$

After averaging (5) and applying (9) to the condition (13), we have

$$\begin{aligned}&\int\limits_{0+}^{t}\int\limits_{0}^{l}\alpha(T)\frac{U(z,t)e^{\mu z+\eta(t)t}-T_c}{v}dzdt = \\ &= c\rho_n\int\limits_{0+}^{t_0}\int\limits_{0}^{l} f_{12}(t)(U(z,t)e^{\mu z+\eta(t)t}-T_0)dzdt.\end{aligned} \tag{14}$$

Based on Eq. (14) we introduce the quadratic residue cost functional [9, 10]

$$J(f)=\int\limits_{\varepsilon+}^{t_0}\int\limits_{0}^{l}\left[W_1(U,f)-W_2(U)\right]^2 dzdt, \tag{15}$$

$$W_1(U,f)=a\frac{U(z,t)e^{\mu z+\eta(t)t}-T_c}{v},$$

$$W_2(U)=f_{12}(t)(U(z,t)e^{\mu z+\eta(t)t}-T_0).$$

In this work we have used Alifanov's iterative regularization method. We complete the statement of an inverse problem by adding initial and boundary conditions to the heat equation. We have the inverse problem (10)–(12) in the area $\bar{Q}_T=\{(z,t)|\, z\in(0,l), t\in(0,t_0)\}$. The function $f(z,t)$ in (10) is found under the condition of a minimum of quadratic residual functional and restriction $J(f)\ge\delta^2$,

$$\delta^2=\int\limits_{0+}^{t_0}\int\limits_{0}^{l}\sigma^2 dzdt,$$

where $\sigma^2$ is dispersion function $W_2(v)$ [10, 11].

The iterative process is built in space $L_2(Q)$, $Q=\Omega\times[0,\tau_m]$ using conjugate gradient method [9–11].

$$f^{k+1}=f^k-\beta_k S^k, k=0,1,...,\bar{k}, \tag{16}$$

where

$$S^k=J_f'^k+\gamma_k S^{k-1}, \gamma_0=0, \quad \gamma_k=\left\|J_f'^k\right\|^2/\left\|J_f'^{k-1}\right\|^2,$$

$$\beta_k=\frac{(U^k-f,V^k)}{\left\|V^k\right\|^2},$$

$U^k=U(f^k,z,\tau)$ – the temperature field at the $k$-th iteration, $V^k=V(\Delta f^k,z,\tau)$ – the gradient of the temperature field at the $k$-th iteration, when the source varies by amount $\Delta f^k$; $(u,w)=\iint\limits_{Q}u(z,t)w(z,t)dzdt$, $\|u\|=\sqrt{(u,u)}$ – scalar product of elements $u(z,t), w(z,t)$ and the norm of element $u$ in space $L_2(Q)$. The gradient of temperature field $V^k$ is obtained from the solution of the homogeneous boundary value problem for the inhomogeneous equation

$$\frac{\partial V}{\partial t}=a^2\frac{\partial^2 V}{\partial^2 z}+\Delta f(z,t), \quad (z,t)\in Q_T \tag{17}$$

$$V(z,0)=0, \quad 0\le z\le l \tag{18}$$

$$V(0,t)=0, \quad V(l,t)=0, \quad 0\le t\le t_0. \tag{19}$$

We get the objective functional gradient using the conjugate variable $\psi(z,t)$. Identity $(Lv,\psi)=(v,L*\psi)$ allows us to write the conditions of the conjugated problem:

$$L*\psi = \zeta, \quad (z,t) \in Q \tag{20}$$

$$\psi(z,t_m) = 0, \tag{21}$$

$$\psi|_{\partial\Omega} = 0, \tag{22}$$

where $L*\psi = \psi_t + a^2\psi_{zz}$, $\zeta = \zeta(z,t)-$ some function. The formula for the gradient can be written as $J'_q = \psi$, $(z,t) \in Q$.

For the organization of the iterative process (16) at each step we calculate the temperature, temperature gradient and the conjugate variable. To find $U(z,t), V(z,t), \psi(z,t)$ we need to solve all three problems (10)–(12), (17)–(19), and (20)–(22). To this end we use a six-point difference scheme [10-18]. Let us introduce the grid $\bar{\omega}_h = \{z_i = ih, i = 0,1,...N\}$, $\omega_\tau = \{t_j = j\tau, j = 0,1..., j_0\}$ and the grid

$\varpi_{h\tau} = \varpi_h \times \omega_\tau = \{(ih, j\tau), i = 0,1,...N, j = 0,1,...j_0\}$ with steps $h = \dfrac{1}{N}$, $\tau = \dfrac{t_0}{j_0}$. Let us denote by $y_i^j$ the values in nodes $(z_i, t_j)$ of grid function $U$, which is defined on $\varpi_{h\tau}$. We replace the derivative $\dfrac{\partial U}{\partial t}$ by a first difference derivative, and the derivative $\dfrac{\partial^2 U}{\partial^2 z}$ by a second difference derivative $U_{\bar{z}z}$. Then we will enter arbitrary real parameter $\sigma$ and consider one-parameter family of difference schemes, *i.e.*,

$$\frac{y_i^{j+1} - y_i^j}{\tau} = \Lambda\left(\sigma y_i^{j+1} + (1-\sigma) y_i^j\right) + \varphi_i^j, \tag{23}$$

$$0 < i < N, 0 \le j < j_0$$

$$y_0^j = T_1^j, \quad y_N^j = T_2^j, \; y_i^0 = y(z_i, 0) = T(z_i),$$

$$\varphi_i^j = f(z_i, t_{j+0,5}), \quad t_{j+0,5} = t_j + 0,5\tau$$

or

$$\varphi_i^j = 0,5(\bar{f} + f), \; \bar{f} = f(z_i, t_{j+0,5}),$$

$$\Lambda y_i = y_{\bar{z}z,i} = (y_{i-1} - 2y_i + y_{i+1}) / h^2.$$

The difference scheme (23) is written in a six-point pattern. At $\sigma = 0.5$ it is absolutely stable with respect to the initial data and the right hand side [10]. Then, the sweep method is used for solving the system of equations. Alifanov's iterative regularization method is a powerful approach for function estimation inverse problems because regularization is implicitly built into the algorithm [10]. The combination of a minimization algorithm with an adjoint equation that provides the gradient to be used in the minimization iterative procedure is the basis of Alifanov's iterative regularization method. In the application of this method several steps have to be considered: the direct, sensitivity and adjoint problems, the gradient equation determination, the conjugate gradient method of minimization and stopping criterion.

CONCLUSIONS. A mathematical model of thermal processes in a cylindrical area with the existing external heat sources is considered in the paper. Methods of solution of inverse boundary value problems for the heat equation are researched. From a physical point of view of the problem thermal processes during thermal cycling and electric pulse treatment of cylindrical shaped samples are described.

According to the type of the inverse problem and the known parameters of the process that describes the mathematical model, the appropriate method of its solution is proposed. In particular the problem of restoration the right side of a parabolic equation is considered. It is shown that the problem of finding the temperature field and external heat sources can be reduced to external and solved by Alifanov's iterative regularization method.

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## МАТЕМАТИЧСКАЯ МОДЕЛЬ ТЕПЛОВОГО ПРОЦЕСА С НЕИЗВЕСТНОЙ ФУНКЦИЕЙ ИСТОЧНИКА

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Математические модели, которые используются при изучении физических и технологических процессов с применением методов компьютерной математики являются мощным инструментом современных исследований. В данной роботе, на примере математической модели теплового процесса, который происходит в движущейся среде предлагается обратная задача, решение которой позволяет определить температурное распределение и плотность внешних источников тепла. Источники тепла представлены в виде финитной функции в краевых условиях. Обратная задача рассмотрена в экстремальной постановке. Для построения целевого квадратического функционала использовалось условие теплового баланса в интегральной форме. Это же условие является дополнительной информацией в обратной задаче теплопроводности. Вводится критерий невязки и строится итерационный процесс по методу сопряженных градиентов. Задача решается численным методом путем построения абсолютно устойчивой, по начальным условиям и правой части шеститочечной разностной схемы. Для решения системы уравнений использован метод прогонки.

**Ключевые слова:** математическая модель, обратная задача, источник тепла, условие теплового баланса.

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