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MATHEMATICAL MODEL OF THE HUMAN OCULOMOTOR SYSTEM IN NORM

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Purpose. To be able to determine the points forces the eye muscles need to create a mathematical model of human oculomotor system, which would include both a primary position in eye sight. **Methodology**. We used mathematical modeling action points forces the eye muscles of both eyes around their axes corresponding three-dimensional rectangular Cartesian coordinate system. **Results**. We proposed a mathematical model of man's oculomotor system, comprising both eyes. The model allows: to determine the coordinates of objects on the surface of both eyeballs as in the primary position of gaze, and at rotation eyeballs within 30° around each of the axes of rectangular three-dimensional Cartesian coordinate system; determine the torques acting on the surface of both eyeballs at the points of attachment of the eye muscles, as in the primary position of gaze, and the rotation of the eyeballs within 30° around each of the axes rectangular three-dimensional Cartesian coordinate system. **Originality**. A mathematical model is a simulation of the options of action on the surface of the eye muscles eyeballs right and left eye in the primary position and in terms of their rotation. **Practical value**. This model makes it possible to determine the effect of the rotation moments of forces in the eyes and explains the nature of these movements from the point of view of Ophthalmology. References 10, tables 2, figures 9.

Key words: oculomotor system, mathematical model, torque, angle of rotation of the eyeball.

МАТЕМАТИЧНА МОДЕЛЬ ОКОРУХОВОГО АПАРАТУ ЛЮДИНИ В НОРМІ

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Запропоновано математичну модель окорухового апарату людини, що включає обидва ока. Модель дозволяє: визначити координати об'єктів на поверхні обох очних яблук, як в первинній позиції погляду, так і при обертанні очних яблук в межах 30° навколо кожної з осей прямокутної декартової тривимірної системи координат; визначити моменти сил що діють на поверхню обох очних яблук в точках прикріплення окорухових м'язів, як в первинній позиції погляду, так і при обертанні очних яблук в межах 30° навколо кожної з осей прямокутної декартової тривимірної системи координат. Створена математична модель є однією з можливих варіантів моделювання дії окорухових м'язів на поверхні очних яблук правого та лівого ока в первинній позиції погляду та при їх обертанні. Така модель дає можливість визначити дію моментів сил при обертанні очей і пояснити при цьому сутність цих рухів с точки зору офтальмології.

Ключові слова: окоруховий апарат, математична модель, моменти сил, кут повороту очного яблука.

PROBLEM STATEMENT. The attempts of modeling of oculomotor system of man had been done since the middle of XIX century. However, substantial progress was only last two or three decades due to computer technique improvement. But a problem is still far from solving. If «in general» nature of the oculomotor system of man has been described by efforts of several authors [1, 2, 3], there remains a number of points where the proposed mathematical models are «stuck». For example, if horizontal lines effect was described quite adequately, the effect of oblique muscles was described «indirectly», as if the last are attached to the sclera at one point, but have not had extended insertion zone in reality. It does not take into account that different parts of the oblique muscles work on the eyeball widely different. On the other hand, the use of offered mathematical models, in work with the certain patients encounters a number of difficulties associated with the inability of obtaining a number of data which are necessary for introduction into the model for practical usage of the last one.

Currently, there is a mathematical model of the impact of the eye muscles on the man's eyeball that allows pre-calculating the coordinates of the points of muscles 'attachment and determining the torques acting on the surface of the eyeball at the points of eye muscles attachment. This model includes an isolated right eye. Creating a similar model, which includes both eyes - is an urgent task.

The aim of this work is to create a mathematical model of human oculomotor system, comprising both eyes.

EXPERIMENTAL PART AND RESULTS OBTAINED. Eyeball is considered as a sphere. Areas of eye muscles attach to the eyeball are considered as lingering objects. Eye muscles traction are treated as vectors emanating from the points of attachment and directed tangent to eyeball at certain angles to the cutting plane passing through the axis OY and OZ three-dimensional Cartesian rectangular system of coordinates. In parallel, the following systems of coordinates are considered: three-dimensional Cartesian rectangular $OXYZ$, axis OY which coincides with the eye sagittal axis with related ophtalmografic spherical system of coordinates (OSSK), its the center of rotation coincides with the center of the Cartesian system of coordinates. Coordinates OSSK are set by two values: the length θ (0-360°) and latitude φ (front 0-90°, rear 0–(-90°). When rotating the eyes $OXYZ$ and OSSK remain motionless. Similar systems $OX_1Y_1Z_1$ and OSSK_1– has right eye (Fig. 1) and $OX_2Y_2Z_2$ and

OSSK$_2$– left eye (Fig. 2) rigidly connected with the respective eyes and rotate with them [4, 5, 6, 7, 8, 9, 10], and the systems of the left eye $OX_2Y_2Z_2$ and OSSK$_2$ are a mirror images of the right eye corresponding $OX_1Y_1Z_1$ and OSSK$_1$. Coordinates of the eye muscles attachment in the norm have been known. In our model *OXYZ* and OSSK are in the primary position of the view it means that axis OY_1 and OY_2 are parallel. Using the created map of the eyeball surface spherical coordinates of eye muscles attaching for the right eyeball and left respectively, are defined.

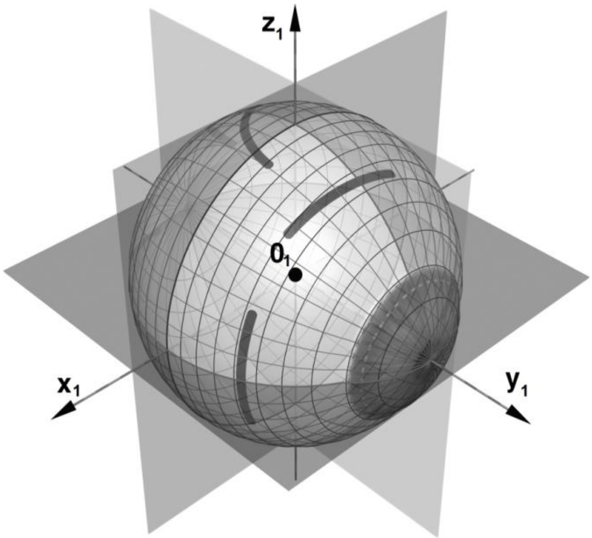

Figure 1 – Image of right eye model in the primary position and look from $OX_1Y_1Z_1$ and OSSK$_1$

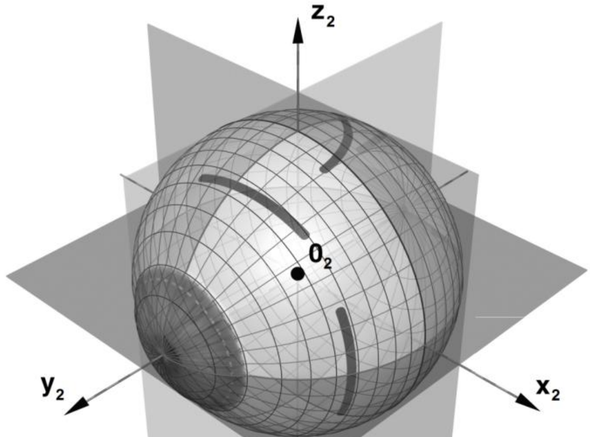

Figure 2 – Image of left eye model in the primary position and look from $OX_2Y_2Z_2$ and OSSK$_2$

Axis OY_1 and OY_2, OZ_1 and OZ_2 match in the directions of ICU, because in our model distance between the centers of rotation of the right and left eye, between the point O_1 and $O_2$is assumed to be the distance between the centers of eyeballs –*dpp*. To determine the relative position coordinates of the left and right eye each other, use the following formula:

$$x_2 = x_1 + dpp, \tag{1}$$

where x_2 – Cartesian coordinate system of the left eyecoordinates; x_1 – Cartesian coordinate system the right eye; *dpp* – the distance between the centers of eyeballs.

Therefore, this expression creates a new point, so-called O_2, center of which coincides with the center of the left eye moving system of coordinates (Fig. 3).

When turning right and left eye coordinates of muscles attachment in the fixed system of coordinates are defined by the formula 2 and 3 respectively:

$$\begin{pmatrix} x_1' \\ y_1' \\ z_1' \end{pmatrix} = A_1 \cdot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}; \tag{2}$$

$$\begin{pmatrix} x_2' \\ y_2' \\ z_2' \end{pmatrix} = A_2 \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \tag{3}$$

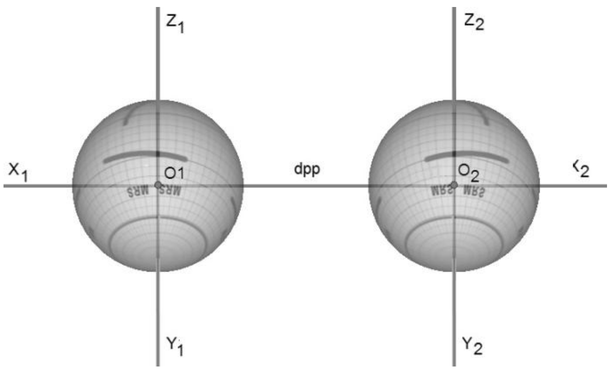

Figure 3 – Model of Image of oculomotor system in primary position and look from *OXYZ* and OSSK

where A_1 and A_2– are transition matrix to new coordinates on the surface of the right and left eyes, respectively.

Type of matrix depends on how rotation is determined. New vector coordinates of action forces of the eye muscles in the fixed and mobile systems of coordinates are defined by using matrices A_1 and A_2 [4]:

$$\bar{F}i_1' = A_1 \cdot \bar{F}i_1; \tag{4}$$

$$\bar{F}i_1' = A_1^{-1} \cdot \bar{F}i_1; \tag{5}$$

$$\bar{F}i_2{}' = A_2 \cdot \bar{F}i_2; \quad (6)$$

$$\bar{F}i_2{}' = A_2^{-1} \cdot \bar{F}i_2, \quad (7)$$

where, A_1^{-1} and A_2^{-1} – matrix inverse to matrices A_1 and A_2 respectively.

The action of each eye muscles in the primary position can be described in terms of torque tension vectors relatively to each of the axes describing vector product of these forces on the radius vector of the left and right eye using matrix respectively:

$$M_{O1} = \bar{r}_1 \times \bar{F}i_1 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_1 & y_2 & z_3 \\ Fx_1 & Fy_1 & Fz_1 \end{vmatrix} =$$

$$= \bar{i} \cdot (y_1 \cdot Fz_1 - z_1 \cdot Fy_1) + \bar{j} \cdot (z_1 \cdot Fx_1 - x_1 \cdot Fz_1) +$$

$$+\bar{k} \cdot (x_1 \cdot Fy_1 - y_1 \cdot Fx_1); \quad (8)$$

$$M_{O2} = \bar{r}_2 \times \bar{F}i_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_2 & y_2 & z_2 \\ Fx_2 & Fy_2 & Fz_2 \end{vmatrix} =$$

$$= \bar{i} \cdot (y_2 \cdot Fz_2 - z_2 \cdot Fy_2) + \bar{j} \cdot (z_2 \cdot Fx_2 - x_2 \cdot Fz_2) +$$

$$+\bar{k} \cdot (x_2 \cdot Fy_2 - y_2 \cdot Fx_2), \quad (9)$$

where, M_{O1}, M_{O2}– are torques actions of oculomotor muscles in the center of right and the left eyeball, respectively; $\bar{r}_1$ and $\bar{r}_2$ – Radius vector of the right and left eyeball, respectively; $\bar{F}i_1$ and $\bar{F}i_2$ – is thrust vector of attachment point of the right and left eyes first muscle respectively; x, y, z– Cartesian coordinates of the muscle surface model of the oculomotor system; Fx, Fy, Fz – projection of thrust vector of i muscle on each axis of three-dimensional Cartesian coordinate system.

Therefore, based on expressions 8 and 9 got a system of equations to determine the points of forces on the right and left eyeballs model surface :

$$\begin{cases} Mx_{i1} = Fz_{i1} \cdot y_{i1} - Fy_{i1} \cdot z_{i1}; \\ My_{i1} = Fx_{i1} \cdot z_{i1} - Fz_{i1} \cdot x_{i1}; \\ Mz_{i1} = Fy_{i1} \cdot x_{i1} - Fx_{i1} \cdot y_{i1}; \\ Mx_{i2} = Fz_{i2} \cdot y_{i2} - Fy_{i2} \cdot z_{i2}; \\ My_{i2} = Fx_{i2} \cdot z_{i2} - Fz_{i2} \cdot x_{i2}; \\ Mz_{i2} = Fy_{i2} \cdot x_{i2} - Fx_{i2} \cdot y_{i2}, \end{cases} \quad (10)$$

where, Fx_{i1}, Fy_{i1}, Fz_{i1}, Fx_{i2}, Fy_{i2}, Fz_{i2}– are projections of thrust vector acting in the i-th point on the axes OX, OY and OZ respectively to each of the eyeballs; Mx_{i1}, My_{i1}, Mz_{i1}, Mx_{i2}, My_{i2}, Mz_{i2}– moments of thrust vectors force regarding to respective axes after the rotation.

For example, let's consider the effect of the thrust vector of MRS_1 muscle (Fig. 4).

This vector, which operates on a point of attachment with the corresponding M_1 of coordinates x_1, y_1 and z_1 right eye rotates clockwise around the axis OX_1 clockwise about its direction. From this we can conclude that the moment of the Mx_{MRS1} force, according to the rules of signs depending on the direction vector of the point will have «+» sign, as the trend of the force directions coincides $OX_{1\ \mathrm{direction}}$, around axis rotation occurs. For the left eye this rotation is in much the same way, i.e. rotation is made around the axis OX_2. But as the direction of the axis OX_2 is opposite to the axis OX_1, the left eye rotates around the axis OX_2 counterclockwise (Fig. 5). Therefore, there is a rule of signs for each such moment of force according to each of the axes (Table 1 and 2).

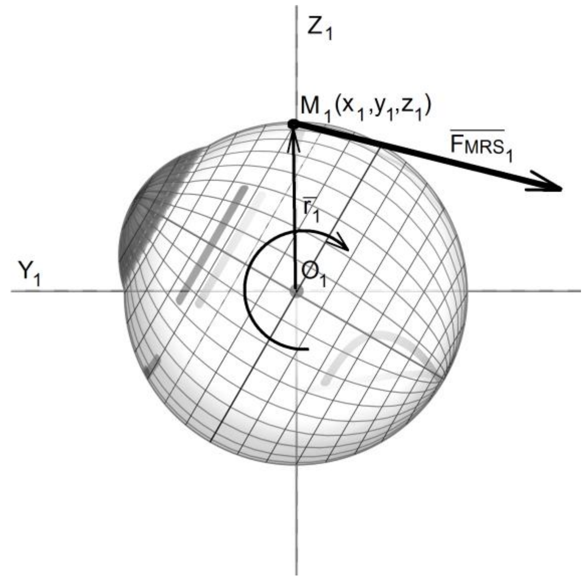

Figure 4 – The right eye rotation around the axis OX_1

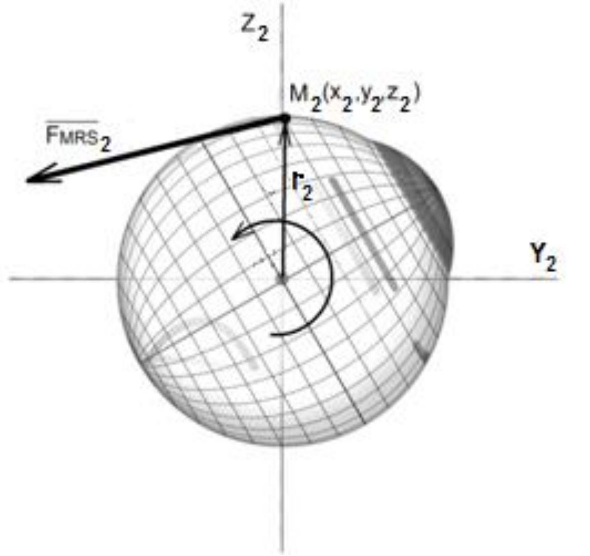

Figure 5 – The left eye rotation around the axis OX_2

Table 1 – Rule marks for moments of forces thrust vector eye muscles right eye

Axis	Turn	Rotation type	Sign
OX_1	clockwise	Lifting	+
OY_1		Excyclduction	+
OZ_1		Bringing	+
OX_1	Counter clockwise	Dropping	-
OY_1		Incyclduction	-
OZ_1		Withdrawal	-

Table 2 – Rule marks for moments of forces thrust vector eye muscles left eye

Axis	Turn	Rotation type	Знак
OX_2	clockwise	Lifting	+
OY_2		Excyclduction	+
OZ_2		Bringing	+
OX_2	clockwise	Dropping	-
OY_2		Incyclduction	-
OZ_2		Withdrawal	-

Rotation of new spherical coordinates of muscle attachment on the the eyeball surface is prsented by formula 11 (for the right eye) and 12 (for the left eye):

$$Q_1\left(\theta_1',\varphi_1'\right) = A_1 \cdot Q_1\left(\theta_1,\varphi_1\right); \quad (11)$$

$$Q_2\left(\theta_2',\varphi_2'\right) = A_2 \cdot Q_2\left(\theta_2,\varphi_2\right). \quad (12)$$

If the coordinate of transformation during rotation of the right eye is known [4], it is possible to determine the transformation coordinate during rotation of the left eye.

Assume that the rotation of the eyeball about an angle δ_2 of the axis OZ_2, and then around the axis OX_2', in which the axis OX_2moves after the rotation about an angle δ_2–about an angle η_2 and finally about the angleμ_2 around OY_2''axis– an image of whichis OY_2 after two axis transformation (Fig.6).

That angles δ_2, η_2 and μ_2– are left eye rotation angles around the OX_2, OY_2 and OZ_2 axis respectively. This applies to the right eye – rotation angles δ_1, η_1 and μ_1, and the axis OX_1, OY_1 and OZ_1 [4].

If angle of the left eyeball around each of the axes rotation is known, the matrix A_2 will be determined as the product of matrices of coordinate transformations in relation to each of the axes:

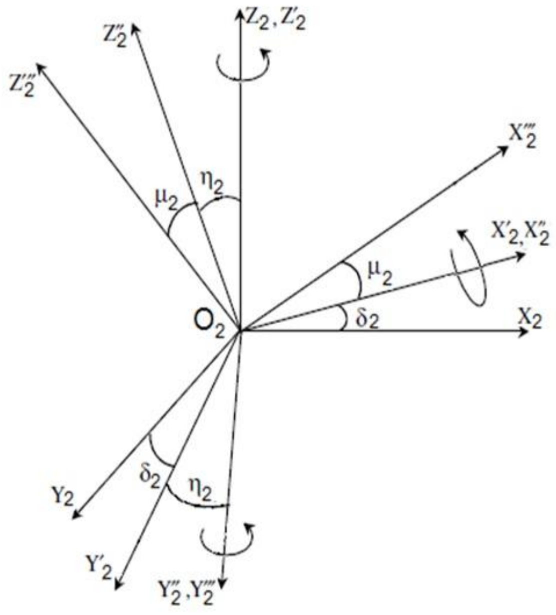

Figure 6 – The rotation of the left eye around the axes

$$A_2 = Ax_2 \cdot Ay_2 \cdot Az_2; \quad (13)$$

$$Ax_2 = \begin{pmatrix} cos\,\eta_2 & 0 & sin\,\eta_2 \\ 0 & 1 & 0 \\ -sin\,\eta_2 & 0 & cos\,\eta_2 \end{pmatrix}; \quad (14)$$

$$Ay_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\,\mu_2 & -sin\,\mu_2 \\ 0 & sin\,\mu_2 & cos\,\mu_2 \end{pmatrix}. \quad (15)$$

$$Az_2 = \begin{pmatrix} cos\,\delta_2 & -sin\,\delta_2 & 0 \\ sin\,\delta_2 & cos\,\delta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad (16)$$

The proposed model adequately operates during the rotation of the eyeballs around each of the axes, as well as the rotation around all axes simultaneously within 30° from the initial position of view. By turning the eyeballs at an angle greater than 30° model in some cases can act inadequately due to the violation of the thrust vector touching rules to the surface of the eyeball.

It should be noted that in some cases where one or two rotations are absent, matrix transformations, which can be obtained from the formula 13 will look like the product of matrices without one or two transformation matrix relatively to the respective axis [4], for example, if the rotation is performed only around OX_2 and OY_2axes, so the matrix A_2 will be:

$$A_2 = Ax_2 \cdot Ay_2. \quad (17)$$

For example, if you know that the left eye has turned around the axis OX_2 and OZ_2 about 10° and 20° respectively, based on the formulas 13, 14 and 16 we have:

$$A_2 = Ax_2 \cdot Az_2 =$$

$$= \begin{pmatrix} \cos\eta_2 & 0 & \sin\eta_2 \\ 0 & 1 & 0 \\ -\sin\eta_2 & 0 & \cos\eta_2 \end{pmatrix} \cdot \begin{pmatrix} \cos\delta_2 & -\sin\delta_2 & 0 \\ \sin\delta_2 & \cos\delta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0{,}925 & -0{,}342 & 0{,}163 \\ 0{,}337 & 0{,}94 & 0{,}059 \\ -0{,}174 & 0 & 0{,}985 \end{pmatrix}.$$

Thus using found matrix A_2there is an ability to find new spherical and Cartesian coordinates of the left eye.

Changing of the points of forces when turning the eyeballs up and down (Fig. 7–9) is clearly demonstrated by the changing of thrust vectors depicted in Figures as segments in the areas of the eye muscles attachment, 3 vectors for each muscle.

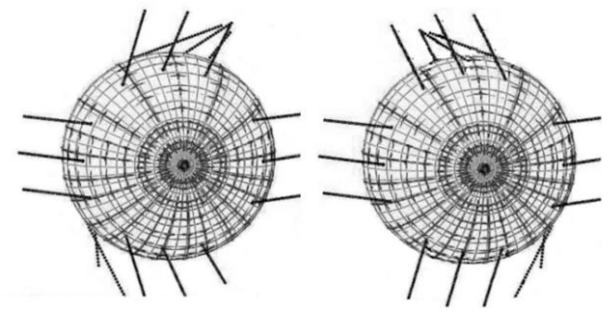

Figure 7 – The primary position of eye sight (both eyes OOSK coincide with $OSSK_1$ and $OSSK_2$)

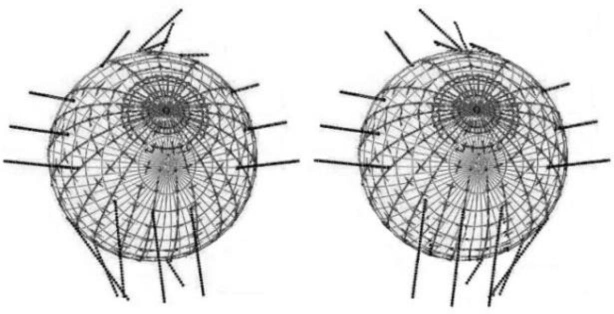

Figure 8 – Looking up the eyes (the ratio OOSK both eyes to $OSSK_1$ and $OSSK_2$ when looking up)

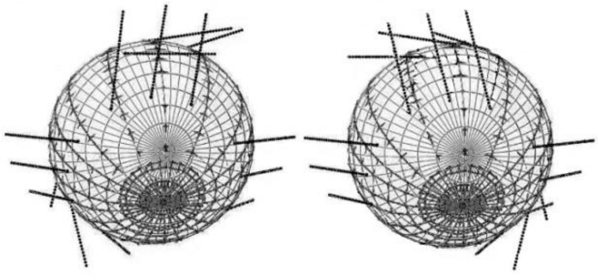

Figure 9 – Looking up the eyes (the ratio OOSK both eyes to $OSSK_1$ and $OSSK_2$ when looking down)

CONCLUSIONS. A mathematical model of the oculomotor system, which includes both eyes has been made. The model allows:

1) To determine the coordinates of objects on the surface of both eyeballs as in the primary position of gaze, and during the rotation of the eyeballs within 30° around each of the axes of the Cartesian three-dimensional coordinate system.

2) To determine the moments of forces acting on the surface of both eyeballs at the points of the eye muscles attachment, as in the primary position of gaze and during the rotation of the eyeballs within 30° around each of the axes of the Cartesian three-dimensional coordinate system.

Thus, this mathematical model is one of the optionalversion of modeling of eye muscles action on the surface of the right and left eyeballs in the primary position of look and during their rotation. This model gives opportunity to determine action of moments of forces during the eye rotation and explains their sense in the context of ophthalmology.

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МАТЕМАТИЧЕСКАЯ МОДЕЛЬ ГЛАЗОДВИГАТЕЛЬНОГО АППАРАТА ЧЕЛОВЕКА В НОРМЕ

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Предложена математическая модель глазодвигательного аппарата, включающего оба глаза. Модель позволяет: определить координаты объектов на поверхности обоих глазных яблок, как в первичной позиции взгляда, так и при повороте глазных яблок в пределах 30° вокруг каждой из осей прямоугольной декартовой трехмерной системы координат; определить моменты сил действующих на поверхность обоих глазных яблок в точках прикрепления глазодвигательных мышц, как в первичной позиции взгляда, так и при обератанни глазных яблок в пределах 30° вокруг каждой из осей прямоугольной декартовой трехмерной системе координат. Созданная математическая модель является одной из возможных вариантов моделирования действия глазодвигательных мышц на поверхности глазных яблок правого и левого глаза в первичной позиции взгляда и при их вращении. Такая модель дает возможность определить действие моментов сил при вращении глаз и объяснить при этом сущность этих движений с точки зрения офтальмологии.

Ключевые слова: глазодвигательный аппарат, математическая модель, моменты сил, угол поворота глазного яблока.

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